Hand in no. 1, 4 and 5 by Nov 9.

Assignment 8

1. Define the operator norm of an $n \times n$-matrix A by

$$\|A\| = \sup\{|Ax| : \ |x| \le 1\} \ ,$$

where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$.

(a) Show that

$$\|A\| = \sup \left\{ \frac{|Ax|}{|x|} : \ x \neq 0 \right\} \ .$$

(b) Show that

$$\|A\| = \inf\{M : \ |Ax| \le M|x|, \ \forall x\} \ .$$

(c) Show that $\|A\|^2$ is equal to the largest eigenvalue of the symmetric matrix A^tA (A^t is the transpose of A).

2. There are other norms defined on $\mathbb{R}^n$ other than the Euclidean one. For example, now consider $\|x\|_1 = \sum_{k=1}^n |x_k|$. For an $n \times n$-matrix A, define

$$\|A\|_1 = \sup\{\|Ax\|_1 : \|x\|_1 \le 1\} \ .$$

(a) Show that

$$\|A\|_1 = \inf\{M : \ \|Ax\|_1 \le M\|x\|_1, \ \forall x\} \ .$$

(b) Show that

$$\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}| \ .$$

(c) Show that the conclusion in Problem 6 in Ex 7 still holds when the condition $\sum_{i,j} a_{ij}^2 < 1$ is replaced by the alternative condition $\|A\|_1 < 1$.

The following problem refreshes your memory in the one dimensional case.

3. Let f be continuously differentiable on $[a, b]$. Show that it has a differentiable inverse if and only if its derivative is not equal to 0 at every point.

4. Consider the function

$$f(x) = \frac{1}{2}x + x^2 \sin \frac{1}{x}, \quad x \neq 0,$$

and set $f(0) = 0$. Show that f is differentiable at 0 with $f'(0) = 1/2$ but it has no local inverse at 0. Does it contradict the Inverse Function Theorem?

5. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x, y) = x - x^2, \ g(x, y) = y + xy$. Show that it has a local inverse at $(0, 0)$. And then write down the inverse map so that its domain can be described explicitly.

6. Let F be a continuously differentiable map from the open $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ whose Jacobian determinant is non-vanishing everywhere. Prove that it maps every open set in U to an open set, that is, F is an open map. Does its inverse $F^{-1} : F(U) \to U$ always exist?